Compensation of non-linear bandwidth broadening by laser chirping in Thomson sources

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A new laser chirping prescription is derived by means of the phase-stationary method for an incident Gaussian laser pulse in conjunction with a Liénard-Wiechert calculation of the scattered radiation flux and spectral brilliance. This particularly efficient laser chirp has been obtained using the electric field of the laser and for electrons and radiation on axis. The frequency modulation is somewhat reduced with respect to that proposed in the previous literature, allowing the application of this procedure to lasers with larger values of the parameter $a_0$. Numerical calculations have been performed using mildly focused and narrow bandwidth laser pulses, confirming a larger efficiency of the chirp prescription here introduced. The chirp efficiency has been analysed as a function of the laser parameter and focusing. Published by AIP Publishing. https://doi.org/10.1063/1.5033549

I. INTRODUCTION

X- and gamma-ray sources with large spectral flux and high tunability allow one to understand the properties of materials and living systems by probing the matter on microscopic-to-atomic and nuclear scales in space and time.1,2 Thomson3–13 sources are among the best performing devices to produce x radiation with narrow bandwidth, high power, and wide tunability, from accelerators with relatively small dimensions and costs. Experiments on the source characterization,14–17 imaging, K-edge techniques, and computed microtomography on phantom,18,19 biological,6,15 animal,11,20,21 and human7,11 samples with keV range x-rays have been already successfully performed. On the other hand, inverse Compton sources, such as the facility Higs22 or the advanced Gamma System ELI-NP-GS,23,24 are aimed at producing extreme gamma ray beams for nuclear physics and nuclear photonics experiments. In these fields, extremely narrow relative bandwidths $\Delta \omega / \omega_0$, down to few $10^{-3}$, are required.

Achieving large spectral fluxes with narrow bandwidth is one of the most demanding challenges in the Thomson/Compton physics. In the linear regime, occurring when the laser parameter $a_0 = eE_0/(mc \omega_L)$ is much smaller than one ($E_0$ and $\omega_L$ being, respectively, the peak electric field and the angular frequency of the laser pulse), the relative bandwidth of the radiation of one single electron collected on axis reflects the relative bandwidth of the laser $\Delta \omega_L / \omega_L$. Starting from this minimum value, many factors contribute to enlarge the bandwidth. First of all, the frequency-angle correlation constitutes an unavoidable limitation connected with the necessity of gathering the radiation within a given acceptance angle. Furthermore, considering the emission from a real electron beam, emittance and energy spread of the particles must be taken carefully under control, in order to diminish their negative effects.25–27 Another source of bandwidth broadening is the non-linear ponderomotive effect that takes place when $a_0$ is of the order or larger than $1.28–30$ The advantage of increasing the luminosity of the source by enhancing the primary photon number is in fact strongly limited by the intrusion in the spectrum of multiple secondary peaks,31–36 due to the interference between wave trains emitted from different positions within the laser pulse. Their number has been empirically estimated as about $N = a_0 \gamma_0^2 \Delta t / 4\sqrt{2\pi}$ ($\Delta t$ is the laser pulse duration).33,37 In addition, when $a_0$ increases, the maximum spectral density on the fundamental peak shifts towards lower frequency values, according to $\omega_{peak} = 4a_0 \gamma_0^2 / (1 + a_0^2 / 2)$, and the spectrum fills the entire region between $\omega_{peak}$ and the linear edge $4a_0 \gamma_0^2$, $\gamma_0$ being the initial Lorentz factor of the electron. In Refs. 37 and 38, a very powerful method for compensating the non-linear bandwidth broadening has been presented. They demonstrated that the presence of a calibrated temporal frequency modulation of the laser—a laser chirp—corrects almost completely the enlargement of the radiation bandwidth due to scattering with a single electron on axis, impeding the formation of oscillations. The physical mechanism is the following: when entering the laser pulse, the electron experiences a varying laser parameter and emits radiation whose frequency varies during the emission process, due to its dependence on the instantaneous electron velocity, on the local laser parameter and on the laser frequency. A temporal variation of the laser frequency along the pulse balances the temporal variation of the other quantities and restores the monochromaticity. In Refs. 37–39, chirp shapes have been calculated and tested, showing a compensation of the non-linear broadening up to laser parameter values close to 1. The same procedure has been advantageously extended to the harmonics40 or to circular polarization.41 Seipt et al.42 proposed a quantum treatment of the problem based on Volkov states, demonstrating that the best chirp does not
depend on quantum recoil and arriving to a chirp prescription similar to that of Ref. 37.

In this paper, we give a laser chirp profile which extends that derived in Ref. 37 in the case of large \(a_0\). The derivation has been done by applying the stationary phase method to the double differential spectrum emitted by one single electron in head-on collision with a Gaussian laser field. The chirp has been calculated for electrons moving and radiation emitted on axis, in the limit of large laser waist. Realistic situations are then analyzed numerically, by adopting a mildly focused and narrow bandwidth laser and collecting the chirped radiation within given acceptance angles \(\theta_{\text{max}}\), thus evaluating the integrated spectrum. The variation of the laser waist \(w_0\) is analyzed. A reduction of the bandwidth from the linear value \((\Delta \omega \omega) / \omega \approx \sqrt{\left(\Delta \omega \omega / \omega \right)^2 + \left(\gamma_0 \theta_{\text{max}} / \omega \right)^2} / 12\) to the limiting value \(\Delta \omega \omega / \omega\) with increasing \(a_0\) is furthermore shown.

II. CHIRP CALCULATION

The calculation has been done in the classical regime, where the quantum recoil is negligible, i.e., when \(h_\omega \gamma_0 / mc^2 \ll 1\). We consider as the primary photon source a Gaussian laser pulse with polarization parallel to the \(x\)-axis, moving along the negative \(z\)-axis with velocity \(c\). Differently from the previous literature, mainly based on the use of the vector potential, we start from the electric field of the pulse, which is given by

\[
\mathcal{E}(x, y, z, t) = \mathcal{E}_0 \mathcal{G} \sin \left(\Gamma - \frac{z}{z_R} \cos \Gamma \right),
\]

where

\[
\mathcal{G} = \frac{1}{1 + \frac{z^2}{z_R^2}} \exp \left( - \frac{z^2}{2 \sigma_{\perp}^2} - \frac{x^2 + y^2}{2 \sigma_{\parallel}^2 (1 + z^2 / z_R^2)} \right).
\]

Here \(\zeta = z + ct, z_R = k_L \sigma_{\perp}^2\) is the Rayleigh length, \(\sigma_{\parallel}\) and \(\sigma_{\perp}\) are the rms longitudinal and transverse dimension of the modulation and \(k_L = 2\pi/\lambda_L\), the carrier wave-number. The function \(\Gamma\) in Eq. (1) is

\[
\Gamma = k_L f(\zeta) + \frac{z z_R (x^2 + y^2)}{2 \sigma_{\perp}^2 (z^2 + z_R^2)},
\]

where

\[
f(\zeta) = \frac{\omega(\zeta)}{\omega_L} = \frac{k(\zeta)}{k_L},
\]

is the frequency modulation giving the local value of the angular frequency \(\omega(\zeta)\) [or wave-number \(k(\zeta)\)] normalized by the carrier frequency \(\omega_L\) (or wave-number \(k_L\)). For a laser pulse without chirp, \(f = 1\). Equation (1) is an approximation of the electric field of the laser pulse solution of the Maxwell equation, in the limit of small \(1/(\sqrt{2} \sigma_{\parallel} k_L)\) and \(1/(\sqrt{2} \sigma_{\perp} k_L)\) and if all modulations are slowly varying functions (the paraxial approximation\(^a\)).

By the same argument, the chirp \(f(\zeta)\) must also be a slowly varying function of the variable \(\zeta\).

For calculating the radiation, we considered the double differential energy spectrum as given in Ref. 44

\[
\frac{d^2 W}{d\omega d\Omega} = \frac{\epsilon^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \frac{n \times (\hat{n} - \hat{\beta}) \times \hat{\beta}}{(1 - n \cdot \hat{\beta})^3} \right|^2.
\]

The laser field expressed by an electron moving close to the axis and in the limit of very large \(\sigma_{\perp}\) reduces to

\[
\mathcal{E}(x, y, z, t) = \mathcal{E}_0 \sin(k_L f(\zeta)) \exp \left( - \frac{\zeta^2}{2 \sigma_{\parallel}^2} \right) = \mathcal{E}_0 \mathcal{g}(\zeta),
\]

the same expression used in the pulsed plane wave model.

The chirp prescription will be calculated on axis.

The electron motion equations in terms of the phase \(\phi(t) = k_L (z(t) + c(t - t_0))\) (\(t\) retarded time) can be written as

\[
\frac{dp_x}{d\phi} = -a_0 g(\phi),
\]

\[
\frac{dp_z}{d\phi} = a_0 g(\phi) \frac{p_t}{\gamma_0},
\]

with \(a_0 = e \mathcal{E}_0 / (\omega_L mc)\), leading to the relations

\[
p_x = - \frac{\gamma_0}{\gamma_0} \int g(\phi') d\phi',
\]

\[
p_z = p_{z_0} - \frac{a_0^2}{2 \gamma_0} \left( \int g(\phi') d\phi' \right),
\]

where \(p_{z_0}\) and \(\gamma_0\) is momentum and Lorentz factor, respectively, of the electron before the head-on collision with the laser pulse.

Calculating \(t'\) and \(z(\phi')\) from (9) and (8) and inserting in (5), we can arrive to

\[
\frac{d^2 W}{d\omega d\Omega} = \frac{\epsilon^2 a_0^2 \gamma_0^2 (1 + \beta_{20})^2}{16\pi^2 c}
\]

\[
\times \left| \sum_{n=1}^{\infty} (-)^n \int_{-\infty}^{\infty} \mathcal{Y}(y_n(\zeta)) \mathcal{Y}(y_{n+1}(\zeta)) e^{2\mathcal{b}_n - y^2} \right|^2,
\]

where \(n\) is the harmonic index, \(\mathcal{Y} = \epsilon k_L \zeta = (z + ct) / (\sqrt{2} \sigma_{\parallel})\) (l laboratory time), and \(\beta_{20}\) is the velocity of the electron before the collision.

The argument \(\zeta\) of the Bessel function appearing in Eq. (10) is

\[
\zeta = \frac{\bar{\omega} A(Y)}{2 b(Y)},
\]

where \(A(Y) = a_0^2 e^{-2Y^2} / 2\),

\[
\bar{\omega} = \frac{\omega}{\omega_{\text{lin}}} = \frac{\omega}{\omega_0 \gamma_0 (1 + \beta_{20})^2},
\]

and \(b(Y)\) is the first derivative of the function \(b(Y) = y f(Y), f(Y)\) being the FM of the laser pulse as defined in (4). Finally, the phases \(\Phi_n\) in (10) are
\[ \Phi_n = \bar{\omega} Y + (1 - 2n)b(Y) + \bar{\omega} \int_0^Y dY' \frac{A(Y')}{(b(Y'))^2}. \quad (13) \]

By applying the stationary phase method to estimate the integral in 10 in the limit \( \varepsilon \to 0 \), as described in more details in Ref. 36, and equating to zero the first derivative of \( \Phi_n \) with respect to \( Y \) [the integration variable in Eq. (10)], we obtain

\[ \bar{\omega} = (2n - 1) \frac{(b)^3}{(b)^3 + A(Y)} \quad (n = 1, 2, 3…). \quad (14) \]

The customary procedure\(^ {37,38} \) is to choose the chirp function \( f(Y) \) in such a way that

\[ (b)^3 = p((b)^3 + A(Y)), \quad (15) \]

where \( p > 0 \) is an arbitrary constant. The electron is therefore forced to emit on the frequencies

\[ \omega_n = p(2n - 1)\omega_{lin} \quad (n = 1, 2, 3…). \quad (16) \]

The solution of Eq. (15) is

\[ b = \frac{d}{dY} (Yf(Y)) = s_1 + s_2 + \frac{p}{3}, \quad (17) \]

with

\[ s_{1,2}(Y) = \left[ \frac{p}{2} A(Y) + \frac{p^3}{27} \mp \sqrt{\frac{p^3}{27} A(Y) + \frac{p^2}{4} A^2(Y)} \right]^{1/3}. \quad (18) \]

Note that \( s_1 \) and \( s_2 \) are positive functions of \( Y \) and therefore \( b(Y) \) is also larger than zero allowing the division by it in Eqs. (11), (13), and (14).

The only regular solution of the differential Eq. (17) is

\[ f(Y) = \frac{p}{3} + \frac{1}{Y} \int_0^Y dY' \left( s_1(Y') + s_2(Y') \right). \quad (19) \]

For small values of \( a_0 \), this function becomes

\[ f_0(Y) = p \left( 1 + \sqrt{\frac{\pi}{2}} \frac{a_0}{4p^2Y} \text{erf}\left(\sqrt{2}Y\right) \right), \quad (20) \]

which is the frequency modulation first studied in Ref. 29, whose Eq. (6) is similar to our (17) provided that \( z + ct = \sqrt{2} Y \sigma \) (\( \sigma \) is the longitudinal laser dimension as defined in Ref. 29) and \( p = 1/(1 + a_0^2/2) \), and then retrieved in Ref. 41 in different conditions of polarization. We consider the more general case (19) with \( p = 1 \).

### III. Numerical Calculations

All the simulations have been done by calculating numerically the electron trajectory under the effect of the Gaussian laser pulse described by expression (1), representing a mildly focused and narrow-bandwidth laser. We have then computed the Liénard-Wiechert electromagnetic field radiated in the far region, as well as the energy spectrum detected on the screen.

In the numerical calculation, the laser, with wavelength \( \lambda_L = 0.5 \mu m \) and linearly polarized on the x-axis, has an rms time duration of \( \Delta t = 0.1 \) ps and the electron started with an incident Lorentz factor value of \( \gamma_0 = 60 \). The shape of the spectrum of the radiation on axis for a laser without chirp \((f = 1)\) is presented in Fig. 1 for an rms spot diameter of \( w_0 = 2\sigma_L = 15 \mu m \) and various values of \( a_0 \), showing the development of the system of fringes.

Figure 2 shows the functions \( f(Y) \) (red line) and \( f_0(Y) \) (black line) as a function of \( Y \), for \( a_0 = 0.5 \) and \( a_0 = 1.58 \). Note that Ref. 37 uses a different boundary condition for the frequency modulation function: \( f(0) = 1 \). In the former case, \( f(Y) \) and \( f_0(Y) \) can hardly be distinguished on the scale of the figure; while enhancing \( a_0 \) they differ increasingly.
substantial difference between Eq. (19) and all other chirp prescriptions proposed in the literature is the trend with \( a_0 \) in the limit of large \( a_0 \). In fact, the limit of Eq. (19) for \( a_0 \gg 1 \) turns out to be

\[
f \approx a_0^{2/3},
\]

(21)

while all other chirp prescriptions follow the trend \( a_0^2 \).

This is connected to the use of the electric field in the motion equations. In fact, from (9), the longitudinal electron velocity can be evaluated approximately as

\[
\beta_z \approx \beta_{z0} - \frac{a_0^2}{2f} \frac{\gamma_0}{f^2}.
\]

(22)

where \( \int_{\Delta \theta} g(\phi')d\phi' \) has been integrated by parts and calculated on the peak. Inserting (22) in the Doppler expression, we obtain

\[
\omega = \frac{\gamma_0 f (1 + \beta_z)}{1 - \beta_z} \approx \frac{2\gamma_0 f}{1 - \beta_z + \frac{a_0^2}{2f} \frac{\gamma_0}{f^2}} = 4\gamma_0^2 \omega_0 \left(1 + \frac{a_0}{f} \frac{\gamma_0}{f^2}\right).
\]

The dependence \( f \sim a_0^{2/3} \) for \( a_0 \) large can be then retrieved by equating

\[
\frac{f}{1 + \frac{a_0}{f} \frac{\gamma_0}{f^2}} = 1
\]

and results somewhat weaker with respect to the dependence on \( a_0^2 \).

Two cases, relevant to \( a_0 = 0.5 \) and \( a_0 = 1 \), are presented, respectively, in Figs. 3 and 4, showing the spectrum on axis in the far field.

In both figures, cases without chirp (in black) are compared with the same cases with the chirp given by \( f_0 \) (in red) and \( f \) (in blue).

FIG. 3. Spectrum of the radiation on axis vs \( \omega/\omega_{lin} \) for \( a_0 = 0.5 \). Black curve: no chirp, number of oscillations \( N = 9.3 \). Red curve: chirp \( f_0 \). Blue curve: chirp \( f \).

FIG. 4. Spectrum of the radiation on axis vs \( \omega/\omega_{lin} \) for \( a_0 = 1 \). Black curve: no chirp, number of oscillations \( N = 37.5 \). Red curve: chirp \( f_0 \). Blue curve: chirp \( f \).

Without the chirp, the spectrum presents the well-known oscillations, shown also in Fig. 1. The effect of the chirp is that of eliminating the ripples and shrinking the bandwidth. When \( a_0 = 0.5 \), (Fig. 3) the two expressions of the chirp give almost the same result, but for \( a_0 = 1 \) (Fig. 4), the expression \( f_0 \) hardly compensates the nonlinear broadening of the spectrum, while \( f \) is amply efficient. The chirp \( f \) is able to make monochromatic also radiation produced by high intensity lasers (with for instance \( a_0 = 10 \)), as shown in Fig. 5.

Since the detector or the sample in an experiment receive the radiation integrated over a surface, we have analyzed the spectrum of the signal collected within a solid acceptance angle (Figs. 6 and 7).

Figures 6(a) and 7(a) show the spectrum for an acceptance angle \( \theta_{max} = 2 \) mrad and \( \theta_{max} = 4 \) mrad for various values of \( a_0 \) without chirp. For the cases with larger \( a_0 \), the radiation on the harmonics produces the tails at the right.

FIG. 5. Spectrum of the radiation on axis vs \( \omega/\omega_{lin} \) for \( a_0 = 10 \). Black curve: no chirp. Red curve: chirp \( f_0 \). Blue curve: chirp \( f \).
The same cases, but with the chirp \( f \), are presented in Figs. 6(b) and 7(b). Since the chirp has been chosen for compensating the broadening of the spectrum on axis, it acts less efficiently off-axis, and therefore a tail remains towards the low frequencies.

Figure 8 presents, at fixed value of \( a_0 \), the comparison of the spectrum on axis obtained with different values of \( w_0 \), for both chirp prescriptions \( f_0 \) [window (a)] and \( f \) [window (b)]. For large values of \( w_0 \), the results are independent on the laser waist, and the plane wave model could be used. However, for decreasing \( w_0 \) (in this case \( w_0 \approx 15 \mu m \)), the gradients of the laser transverse distribution begin to affect the electron trajectories, and the chirp loses progressively efficiency. Contrarily to \( f_0 \), \( f \) is able to correct the nonlinear broadening for more focused laser pulses, down to \( w_0 = 5 \mu m \). For smaller waists, the paraxial model is no more valid and a full wave integration of the Maxwell equations is needed.

Finally, in Fig. 9, we have compared, for the acceptance angles \( \theta_{max} = 2 \text{ mrad} \) and \( \theta_{max} = 4 \text{ mrad} \), the relative rms bandwidth \( \Delta \omega / \omega \) on the fundamental frequency with and without chirp as a function of the laser parameter \( a_0 \).

In the case of a laser without chirp (empty stars), the radiation relative bandwidth strongly increases with \( a_0 \) due to the non-linear broadening. In the case of a laser with chirp (full stars), instead, the relative bandwidth decreases from the linear prevision \( (\Delta \omega / \omega)_{\text{acc-LIM}} \approx \sqrt{(\Delta \omega_{L} / \omega_{L})^2 + (\gamma_0 \theta_{max})^2 / 12} \) (green and dark green lines) to the laser bandwidth \( \Delta \omega_{L} / \omega_{L} \) (yellow line). Even if the chirp is chosen in order to compensate the broadening on axis and is therefore less efficient off-axis, the bandwidth of the radiation collected within \( \theta_{max} = 2 \text{ mrad} \) remains narrow due to the high and pronounced radiation intensity peak occurring in correspondence of the axis of the system. The present calculations are done in the case of a laser with Gaussian transverse and longitudinal profiles. The pulse is furthermore assumed transform limited. Deviations from this ideal case introduce other spectral broadenings with the effect of increasing the bandwidth.

**IV. CONCLUSIONS**

We have derived a new expression of the chirp prescription that corrects the non-linear broadening on fundamental peak and harmonics up to laser pulse parameters well larger than 1, expanding and strengthening the result of Refs. 37 and 38. Realistic three dimensional situations are analyzed,
The robustness of the method will be therefore tested. The within given acceptance angles and evaluating the spectrum. using a focused laser pulse, collecting the chirped radiation within given acceptance angles and evaluating the spectrum. The results demonstrate that increasing the laser parameter, the chirp permits one to decrease the rms bandwidth under the linear value, up to two orders of magnitude lower than the nonchirped case.

FIG. 8. Spectrum of the radiation on axis vs \(\omega/\omega_{\text{lin}}\) for \(\theta_0 = 0.6\) and \(w_0 = 70\) \(\mu\text{m}\) (blue line), \(w_0 = 20\) \(\mu\text{m}\) (blue line), \(w_0 = 7\) \(\mu\text{m}\) (magenta line), \(w_0 = 5\) \(\mu\text{m}\) (red line) for both chirp prescriptions \(f_0\) [window (a)] and \(f\) [window (b)].

FIG. 9. Relative rms radiation bandwidth \(\Delta\omega/\omega\) vs \(\theta_0\) for \(\theta_{\text{max}} = 2\) mrad (blue) and \(\theta_{\text{max}} = 4\) mrad (red). Empty stars: without chirp. Full stars: with chirp \(f\). On the plot, the laser bandwidth \(\Delta\omega/\omega_{\text{lin}}\) (yellow line) and the estimated linear bandwidths for the two acceptance angles, respectively, \((\Delta\omega/\omega)_{\text{acc}+1.2\text{mrad}}\) (light green) and \((\Delta\omega/\omega)_{\text{acc}+1.2\text{mrad}}\) (dark green) are also shown.